

Real-Time Sequential Convex Programming for Optimal Control Applications

Tran Dinh Quoc[†], Carlo Savorgnan[†] and Moritz Diehl[†]

[†] Department of Electrical Engineering (ESAT-SCD) and Optimization in Engineering Center (OPTEC), K.U. Leuven, Kasteelpark Arenberg 10, B-3001 Leuven, Belgium
{quoc.trandinh, carlo.savorgnan, moritz.diehl}@esat.kuleuven.be

Summary. This paper proposes real-time sequential convex programming (RTSCP), a method for solving a sequence of nonlinear optimization problems depending on an online parameter. We provide a contraction estimate for the proposed method and, as a byproduct, a new proof of the local convergence of sequential convex programming. The approach is illustrated by an example where RTSCP is applied to nonlinear model predictive control.

1 Introduction and motivation

Consider a parametric optimization problem of the form:

$$\begin{cases} \min_x c^T x \\ \text{s.t. } g(x) + M\xi = 0, \ x \in \Omega, \end{cases} \quad \mathbf{P}(\xi)$$

where $x, c \in \mathbf{R}^n$, $g : \mathbf{R}^n \rightarrow \mathbf{R}^m$ is a nonlinear function, $\Omega \subseteq \mathbf{R}^n$ is a convex set, the parameter ξ belongs to a given set $\Gamma \subseteq \mathbf{R}^p$, and $M \in \mathbf{R}^{m \times p}$ is a given matrix.

This paper deals with the efficient calculation of approximate solutions to a sequence of problems of the form $\mathbf{P}(\xi)$ where the parameter ξ is varying slowly. In other words, for a sequence $\{\xi_k\}_{k \geq 1}$ such that $\|M(\xi_{k+1} - \xi_k)\|$ is small, we want to solve problem $\mathbf{P}(\xi_k)$ in an efficient way without requiring too much accuracy in the result.

In practice, sequences of problems of the form $\mathbf{P}(\xi)$ can be solved in the framework of nonlinear model predictive control (MPC). MPC is an optimal control technique which avoids computing an optimal control law in a feedback form, which is often a numerically intractable problem. A popular way of solving the optimization problem to calculate the control sequence is using either interior point methods [1] or sequential quadratic programming (SQP) [2, 3, 9]. A drawback of using SQP is that this method may require several iterations before convergence and therefore the computation time may be too large for a real-time implementation. A solution to this problem was proposed in [6], where the real-time iteration (RTI) technique was introduced. Extensions to the original idea and some theoretical results are reported in [5, 7, 8]. Similar nonlinear MPC algorithms are proposed in [10, 13]. RTI is based

on the observation that for several practical applications of nonlinear MPC, the data of two successive optimization problems to be solved in the MPC loop is numerically close. In particular, if we express these optimization problems in the form $P(\xi)$, the parameter ξ usually represents the current state of the system, which, for most applications, doesn't change significantly in two successive measurements. The RTI technique consists of performing only the first step of the usual SQP algorithm which is initialized using the solution calculated in the previous MPC iteration.

Contribution. Before stating the main contributions of the paper we need to outline the (full-step) sequential convex programming (SCP) algorithm framework applied to problem $P(\xi)$ for a given value ξ_k of the parameter ξ :

1. Choose a starting point $x^0 \in \Omega$ and set $j := 0$.
2. Solve the convex approximation of $P(\xi_k)$:

$$\begin{cases} \min_x c^T x \\ \text{s.t. } g'(x^j)(x - x^j) + g(x^j) + M\xi_k = 0, \\ x \in \Omega \end{cases} \quad P_{\text{cvx}}(x^j; \xi_k)$$

to obtain a solution x^{j+1} , where $g'(\cdot)$ is the Jacobian matrix of $g(\cdot)$.

3. If the stopping criterion is satisfied then: STOP. Otherwise, set $j := j + 1$ and go back to Step 2.

The real-time sequential convex programming (RTSCP) method proposed in this paper combines the RTI technique and the SCP algorithm: instead of solving with SCP every $P(\xi_k)$ to full accuracy, RTSCP solves only one convex approximation $P_{\text{cvx}}(x^{k-1}; \xi_k)$ using as a linearization point x^{k-1} , which is the approximate solution of $P(\xi_{k-1})$ calculated at the previous iteration. Therefore, RTSCP solves a sequence of convex problems corresponding to the different problems $P(\xi_k)$. This method is suitable for the problems that contain a general convex substructure such as nonsmooth convex cost, second order or semidefinite cone constraints which may not be convenient for SQP methods.

In this paper we provide a contraction estimate for RTSCP which can be interpreted in the following way: if the linearization of the first problem $P(\xi_0)$ is close enough to the solution of the problem and the quantity $\|M(\xi_{k+1} - \xi_k)\|$ is not too big (which is the case for many problems arising from nonlinear MPC), RTSCP provides a sequence of good approximations of the sequence of optimal solutions of the problems $P(\xi_k)$. As a byproduct of this result, we obtain a new proof of local convergence for the SCP algorithm.

The paper is organized as follows. Section 2 proposes a description of the RTSCP algorithm. Section 3 proves the contraction estimate for the RTSCP method. The last section shows an application of the RTSCP method to nonlinear MPC.

2 The RTSCP method

As mentioned in the previous section, SCP solves a possibly nonconvex optimization problem by solving a sequence of convex subproblems which approximate the orig-

inal problem locally. In this section, we combine RTI and SCP to obtain the RTSCP method. The method consists of the following steps:

Initialization. Find an initial value $\xi_1 \in \Gamma$, choose a starting point $x^0 \in \Omega$ and compute the information needed at the first iteration such as derivatives, dependent variables, Set $k := 1$.

Iteration.

1. Solve $P_{\text{cvx}}(x^{k-1}; \xi_k)$ (see Section 3) to obtain a solution x^k .
2. Determine a new parameter $\xi_{k+1} \in \Gamma$, update (or recompute) the information needed for the next step. Set $k := k + 1$ and go back to Step 1.

One of the main tasks of the RTSCP method is to solve the convex subproblem $P_{\text{cvx}}(x^{k-1}; \xi_k)$ at each iteration. This work can be done by either implementing an optimization method which exploits the problem structure or relying on one of the many efficient software tools available nowadays.

Remark 1. In the RTSCP method, a starting point x^0 in Ω is required. It can be any point in Ω . But as we will show later [Theorem 1], if we choose x^0 close to the true solution of $P(\xi_0)$ and $\|M(\xi_1 - \xi_0)\|$ is sufficiently small, then the solution x^1 of $P_{\text{cvx}}(x^0, \xi_1)$ is still close to the true solution of $P(\xi_1)$. Therefore, in practice, problem $P(\xi_0)$ can be solved approximately to get a starting point x^0 .

Remark 2. Problem $P(\xi)$ has a linear cost function. However, RTSCP can deal directly with the problems where the cost function $f(x)$ is convex. If the cost function is quadratic and Ω is a polyhedral set then the RTSCP method collapses to the real-time iteration of a Gauss-Newton method (see, e.g. [4]).

Remark 3. In MPC, the parameter ξ is usually the value of the state variables of a dynamic system at the current time t . In this case, ξ is measured at each sample time based on the real-world dynamic system (see example in Section 4).

3 RTSCP contraction estimate

The KKT conditions of problem $P(\xi)$ can be written as

$$\begin{cases} 0 \in c + g'(x)^T \lambda + N_\Omega(x) \\ 0 = g(x) + M\xi, \end{cases} \quad (1)$$

where $N_\Omega(x) := \{u \in \mathbf{R}^n \mid u^T(v - x) \geq 0, \forall v \in \Omega\}$ if $x \in \Omega$ and $N_\Omega(x) := \emptyset$ if $x \notin \Omega$, is the normal cone of Ω at x , and λ is a Lagrange multiplier associated with g . Note that the constraint $x \in \Omega$ is implicitly included in the first line of (1). A pair $\bar{z}(\xi) := (\bar{x}(\xi), \bar{\lambda}(\xi))$ satisfying (1) is called a KKT point and $\bar{x}(\xi)$ is called a stationary point of $P(\xi)$. We denote by $A(\xi)$ the set of KKT points at ξ .

In the sequel, we use z for a pair (x, λ) , \bar{z}^k is a KKT point of $P(\xi)$ at ξ_k and z^k is a KKT point of $P_{\text{cvx}}(x^k; \xi_{k+1})$ (defined below) at ξ_{k+1} for $k \geq 0$. The symbols $\|\cdot\|$ and $\|\cdot\|_F$ stand for the L_2 -norm and the Frobenius norm, respectively.

Now, let us define $\varphi(z; \xi) := \begin{pmatrix} c + g'(x)^T \lambda \\ g(x) + M\xi \end{pmatrix}$ and $K := \Omega \times \mathbf{R}^m$, then the KKT system (1) can be expressed as a parametric *generalized equation* [11]:

$$0 \in \varphi(z; \xi) + N_K(z), \quad (2)$$

where $N_K(z)$ is the normal cone of K at z .

Let $x^k \in \Omega$ be a solution of $\mathbf{P}_{\text{cvx}}(x^{k-1}; \xi_k)$ at the k -iteration of RTSCP. We consider the following parametric convex subproblem at Step 1 of the RTSCP algorithm:

$$\begin{cases} \min_x c^T x \\ \text{s.t. } g'(x^k)(x - x^k) + g(x^k) + M\xi_{k+1} = 0, \\ x \in \Omega. \end{cases} \quad \mathbf{P}_{\text{cvx}}(x^k; \xi_{k+1})$$

If we define $\hat{\varphi}(z; x^k, \xi_{k+1}) := \begin{pmatrix} c + g'(x^k)^T \lambda \\ g(x^k) + g'(x^k)(x - x^k) + M\xi_{k+1} \end{pmatrix}$ then the KKT condition for $\mathbf{P}_{\text{cvx}}(x^k, \xi_{k+1})$ can also be represented as a parametric generalized equation:

$$0 \in \hat{\varphi}(z; x^k, \xi_{k+1}) + N_K(z), \quad (3)$$

where $\eta_k := (x^k, \xi_{k+1})$ plays a role of parameter. Suppose that the Slater constraint qualification condition holds for problem $\mathbf{P}_{\text{cvx}}(x^k; \xi_{k+1})$, i.e.:

$$\text{ri}(\Omega) \cap \{x : g(x^k) + g'(x^k)(x - x^k) + M\xi_{k+1} = 0\} \neq \emptyset,$$

where $\text{ri}(\Omega)$ is the set of the relative interior points of Ω . Then by convexity of Ω , a point $z^{k+1} = (x^{k+1}, \lambda^{k+1})$ is a KKT point of the subproblem $\mathbf{P}_{\text{cvx}}(x^k; \xi_{k+1})$ if and only if x^{k+1} is a solution of $\mathbf{P}_{\text{cvx}}(x^k; \xi_{k+1})$ with a corresponding multiplier λ^{k+1} .

For a given KKT point $\bar{z}^k \in \Lambda(\xi_k)$ of $\mathbf{P}(\xi_k)$, we define a set-valued mapping:

$$L(z; \xi) := \hat{\varphi}(z; \bar{x}^k, \xi) + N_K(z), \quad (4)$$

and $L^{-1}(\delta; \xi) := \{z \in \mathbf{R}^{n+m} : \delta \in L(z; \xi)\}$ for $\delta \in \mathbf{R}^{n+m}$ is its inverse mapping. Note that $0 \in L(z; \xi)$ is indeed the KKT condition of $\mathbf{P}_{\text{cvx}}(\bar{x}^k; \xi)$. For each $k \geq 0$, we make the following assumptions:

- (A1) The set of the KKT points $\Lambda_0 := \Lambda(\xi_0)$ is nonempty.
- (A2) The function g is twice continuously differentiable on its domain.
- (A3) There exist a neighborhood $\mathcal{N}_0 \subset \mathbf{R}^{n+m}$ of the origin and a neighborhood $\mathcal{N}_{\bar{z}^k}$ of \bar{z}^k such that for each $\delta \in \mathcal{N}_0$, $\psi_k(\delta) := \mathcal{N}_{\bar{z}^k} \cap L^{-1}(\delta; \xi)$ is single-valued and Lipschitz continuous on \mathcal{N}_0 with a Lipschitz constant $\gamma > 0$.
- (A4) There exists a constant $0 \leq \kappa < 1/\gamma$ such that $\|E_g(\bar{z}^k)\|_F \leq \kappa$, where $E_g(\bar{z}^k) := \sum_{i=1}^m \bar{\lambda}_i^k \nabla^2 g_i(\bar{x}^k)$.

Assumptions (A1) and (A2) are standard in optimization, while Assumption (A3) is related to the *strong regularity* concept introduced by Robinson [11] for the parametric generalized equations of the form (2). It is important to note that the strong regularity assumption follows from the strong second order sufficient optimality in nonlinear programming when the constraint qualification condition (LICQ)

holds [11] [Theorem 4.1]. In this paper, instead of the generalized linear mapping $L_R(z; \xi) := \varphi(\bar{z}^k; \xi) + \varphi'(\bar{z}^k)(z - \bar{z}^k) + N_K(z)$ used in [11] to define strong regularity, in Assumption (A3) we use a similar form $L(z; \xi) = \varphi(\bar{z}^k; \xi) + D(\bar{z}^k)(z - \bar{z}^k) + N_K(z)$, where

$$\varphi'(\bar{z}^k) = \begin{bmatrix} E_g(\bar{z}^k) & g'(\bar{x}^k)^T \\ g'(\bar{x}^k) & 0 \end{bmatrix}, \text{ and } D(\bar{z}^k) = \begin{bmatrix} 0 & g'(\bar{x}^k)^T \\ g'(\bar{x}^k) & 0 \end{bmatrix}.$$

These expressions are different from each other only at the left-top corner $E_g(\tilde{z}^k)$, the Hessian of the Lagrange function. Assumption (A3) corresponds to the standard strong regularity assumption (in the sense of Robinson [11]) of the subproblem $P_{\text{vx}}(x^k; \xi_{k+1})$ at the point \tilde{z}^k , a KKT point of (2) at $\xi = \xi_k$.

Assumption (A4) implies that either the function g should be “weakly nonlinear” (small second derivatives) in a neighborhood of a stationary point or the corresponding Lagrange multipliers are sufficiently small in this neighborhood. The latter case occurs if the optimal value of $P(\xi)$ depends only weakly on perturbations of the nonlinear constraint $g(x) + M\xi = 0$.

Theorem 1 (Contraction Theorem). *Suppose that Assumptions (A1)-(A4) are satisfied. Then there exist neighborhoods \mathcal{N}_τ of ξ_k , \mathcal{N}_ρ of \bar{z}^k and a single-valued function $\bar{z} : \mathcal{N}_\tau \rightarrow \mathcal{N}_\rho$ such that for all $\xi_{k+1} \in \mathcal{N}_\tau$, $\bar{z}^{k+1} := \bar{z}(\xi_{k+1})$ is the unique KKT point of $P(\xi_{k+1})$ in \mathcal{N}_ρ with respect to parameter ξ_{k+1} (i.e. $\Lambda(\xi_{k+1}) \neq \emptyset$). Moreover, for any $\xi_{k+1} \in \mathcal{N}_\tau$, $z^k \in \mathcal{N}_\rho$ we have*

$$\|z^{k+1} - \bar{z}^{k+1}\| \leq \omega_k \|z^k - \bar{z}^k\| + c_k \|M(\xi_{k+1} - \xi_k)\|, \quad (5)$$

where $\omega_k \in (0, 1)$, $c_k > 0$ are constant, and z^{k+1} is a KKT point of $P_{\text{cvx}}(x^k; \xi_{k+1})$.

Proof. The proof is organized in two parts and step by step. The first part proves $\Lambda_k := \Lambda(\xi_k) \neq \emptyset$ for all $k \geq 0$ by induction and estimates the norm $\|\bar{z}^{k+1} - \bar{z}^k\|$. The second part proves the inequality (5).

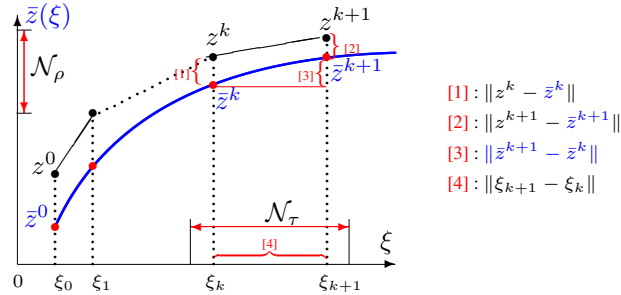


Fig. 1. The approximate sequence $\{z^k\}_k$ along the manifold $\bar{z}(\cdot)$ of the KKT points.

Part 1: For $k = 0$, $\Lambda_0 \neq \emptyset$ by Assumption (A1). Suppose that $\Lambda_k \neq \emptyset$ for $k \geq 0$, we will show that $\Lambda_{k+1} \neq \emptyset$. We divide the proof into four steps.

Step 1.1. We first provide the following estimations. Take any $\bar{z}^k \in \Lambda_k$. We define

$$r_k(z; \xi) := \hat{\varphi}(z; \bar{x}^k, \xi_k) - \varphi(z; \xi). \quad (6)$$

Since $\gamma\kappa < 1$ by (A4), we can choose $\varepsilon > 0$ sufficiently small such that $\gamma\kappa + 5\sqrt{3}\gamma\varepsilon < 1$. By the choice of ε , we also have $c_0 := \kappa + \sqrt{3}\varepsilon \in (0, 1/\gamma)$. Since g is twice continuously differentiable, there exist neighborhoods $\mathcal{N}_\tau \subseteq \mathcal{N}_{\xi_k}$ of ξ_k and $\mathcal{N}_\rho \subseteq \mathcal{N}_{\bar{z}^k}$ of a radius $\rho > 0$ centered at \bar{z}^k such that: $r_k(z; \xi) \in \mathcal{N}_0$, $\|E_g(z) - E_g(\bar{z}^k)\|_F \leq \varepsilon$, $\|E_g(z) - E_g(\bar{z}^k)\|_F \leq \varepsilon$, $\|g'(x) - g'(\bar{x}^k)\|_F \leq \varepsilon$ and $\|g'(x) - g'(\bar{x}^k)\|_F \leq \varepsilon$ for all $z \in \mathcal{N}_\rho$.

Next, we shrink the neighborhood \mathcal{N}_τ of ξ_k , if necessary, such that:

$$\|M(\xi - \xi_k)\| \leq \rho(1 - c_0)/\gamma. \quad (7)$$

Step 1.2. For any $z, z' \in \mathcal{N}_\rho$, we now estimate $\|r_k(z; \xi) - r_k(z'; \xi)\|$. From (6) we have

$$\begin{aligned} r_k(z; \xi) - r_k(z'; \xi) &= \hat{\varphi}(z; \bar{x}^k, \xi_k) - \hat{\varphi}(z'; \bar{x}^k, \xi_k) - \varphi(z; \xi) + \varphi(z'; \xi) \\ &= \int_0^1 B(z_t; \bar{x}^k)(z' - z)dt, \end{aligned} \quad (8)$$

where $z_t := z + t(z' - z) \in \mathcal{N}_\rho$ and

$$B(z; \hat{x}) = \begin{bmatrix} E_g(z) & g'(z)^T - g'(\hat{x})^T \\ g'(x) - g'(\hat{x}) & 0 \end{bmatrix}. \quad (9)$$

Using the estimations of E_g and g' at *Step 1.1*, it follows from (9) that

$$\begin{aligned} \|B(z_t; \bar{x}^k)\| &\leq \|E_g(\bar{z}^k)\|_F + [\|E_g(z_t) - E_g(\bar{z}^k)\|_F^2 + 2\|g'(z_t) - g'(\bar{z}^k)\|_F^2]^{1/2} \\ &\leq \kappa + \sqrt{3}\varepsilon \equiv c_0. \end{aligned} \quad (10)$$

Substituting (10) into (8), we get

$$\|r_k(z; \xi) - r_k(z'; \xi)\| \leq c_0\|z - z'\|. \quad (11)$$

Step 1.3. Let us define $\Phi_\xi(z) := \mathcal{N}_{\bar{z}^k} \cap L(r_k(z; \xi); \xi_k)$. Next, we show that $\Phi_\xi(\cdot)$ is a contraction self-mapping onto \mathcal{N}_ρ and then show that $\Lambda_{k+1} \neq \emptyset$.

Indeed, since $r_k(z; \xi) \in \mathcal{N}_0$, applying (A3) and (11), for any $z, z' \in \mathcal{N}_\rho$, one has

$$\|\Phi_\xi(z) - \Phi_\xi(z')\| \leq \gamma\|r_k(z; \xi) - r_k(z'; \xi)\| \leq \gamma c_0\|z - z'\|. \quad (12)$$

Since $\gamma c_0 \in (0, 1)$ (see *Step 1.1*), we conclude that $\Phi_\xi(\cdot)$ is a contraction mapping on \mathcal{N}_ρ . Moreover, since $\bar{z}^k = \mathcal{N}_{\bar{z}^k} \cap L^{-1}(0; \xi_k)$, it follows from (A3) and (7) that

$$\|\Phi_\xi(\bar{z}^k) - \bar{z}^k\| \leq \gamma\|r_k(\bar{z}^k; \xi)\| = \gamma\|M(\xi - \xi_k)\| \leq (1 - \gamma c_0)\rho.$$

Combining the last inequality, (12) and noting that $\|z - \bar{z}^k\| \leq \rho$ we obtain

$$\|\Phi_\xi(z) - \bar{z}^k\| \leq \|\Phi_\xi(z) - \Phi_\xi(\bar{z}^k)\| + \|\Phi_\xi(\bar{z}^k) - \bar{z}^k\| \leq \rho,$$

which proves Φ_ξ is a self-mapping onto \mathcal{N}_ρ . Consequently, for any $\xi_{k+1} \in \mathcal{N}_\tau$, $\Phi_{\xi_{k+1}}$ possesses a unique fixed point \bar{z}^{k+1} in \mathcal{N}_ρ by virtue of the *contraction principle*. This

statement is equivalent to \bar{z}^{k+1} is a KKT point of $P(\xi_{k+1})$, i.e. $\bar{z}^{k+1} \in \Lambda(\xi_{k+1})$. Hence, $\Lambda_{k+1} \neq \emptyset$.

Step 1.4. Finally, we estimate $\|\bar{z}^{k+1} - \bar{z}^k\|$. From the properties of Φ_ξ we have

$$\|\bar{z}^{k+1} - z\| \leq (1 - c_0\gamma)^{-1} \|\Phi_{\xi_{k+1}}(z) - z\|, \quad \forall z \in \mathcal{N}_\rho. \quad (13)$$

Using this inequality with $z = \bar{z}^k$ and noting that $\bar{z}^k = \Phi_{\xi_k}(\bar{z}^k)$, we have

$$\|\bar{z}^{k+1} - \bar{z}^k\| \leq (1 - c_0\gamma)^{-1} \|\Phi_{\xi_{k+1}}(\bar{z}^k) - \Phi_{\xi_k}(\bar{z}^k)\|. \quad (14)$$

Since $\|r_k(\bar{z}^k; \xi_k) - r_k(\bar{z}^k; \xi_{k+1})\| = \|M(\xi_{k+1} - \xi_k)\|$, applying again (A3), it follows from (14) that

$$\|\bar{z}^{k+1} - \bar{z}^k\| \leq (1 - c_0\gamma)^{-1} \gamma \|M(\xi_{k+1} - \xi_k)\|. \quad (15)$$

Part 2: Let us define the residual from $\hat{\varphi}(z; \bar{x}^k, \xi_{k+1})$ to $\hat{\varphi}(z; x^k, \xi_{k+1})$ as:

$$\delta(z; x^k, \xi_{k+1}) := \hat{\varphi}(z; \bar{x}^k, \xi_{k+1}) - \hat{\varphi}(z; x^k, \xi_{k+1}). \quad (16)$$

Step 2.1. We first provide an estimation for $\|\delta(z; x^k, \xi_{k+1})\|$. From (16) we have

$$\begin{aligned} \delta(z; x^k, \xi_{k+1}) &= [\hat{\varphi}(z; \bar{x}^k, \xi_{k+1}) - \varphi(\bar{z}^k, \xi_{k+1})] - [\varphi(z; \xi_{k+1}) - \varphi(\bar{z}^k, \xi_{k+1})] \\ &\quad - [\hat{\varphi}(z; x^k, \xi_{k+1}) - \varphi(z^k, \xi_{k+1})] + [\varphi(z; \xi_{k+1}) - \varphi(z^k, \xi_{k+1})] \\ &= \int_0^1 B(z_t^k; x^k)(z - z^k)dt - \int_0^1 B(\bar{z}_t^k; \bar{x}^k)(z - \bar{z}^k)dt \\ &= \int_0^1 [B(z_t^k; x^k) - B(\bar{z}_t^k; \bar{x}^k)](z - z^k)dt - \int_0^1 B(\bar{z}_t^k; \bar{x}^k)(z^k - \bar{z}^k)dt, \end{aligned} \quad (17)$$

where $z_t^k := z^k + t(z - z^k)$, $\bar{z}_t^k := \bar{z}^k + t(z - \bar{z}^k)$ and B is defined by (9). Using the definition of $\hat{\varphi}$ and the estimations of E_g and g' at Step 1.1, it is easy to show that

$$\begin{aligned} \|B(z_t^k; x^k) - B(\bar{z}_t^k; \bar{x}^k)\| &\leq [\|E_g(z_t^k) - E_g(\bar{z}_t^k)\|_F^2 + 2\|g'(x_t^k) - g'(\bar{x}^k)\|_F^2]^{1/2} \\ &\quad + [\|E_g(\bar{z}_t^k) - E_g(\bar{z}^k)\|_F^2 + 2\|g'(\bar{x}_t^k) - g'(\bar{x}^k)\|_F^2]^{1/2} \leq 2\sqrt{3}\varepsilon. \end{aligned} \quad (18)$$

Similar to (10), the quantity $B(\bar{z}_t^k; \bar{x}^k)$ is estimated by

$$\|B(\bar{z}_t^k; \bar{x}^k)\| \leq \kappa + \sqrt{3}\varepsilon. \quad (19)$$

Substituting (18) and (19) into (17), we obtain an estimation for $\|\delta(z; x^k, \xi_{k+1})\|$ as

$$\|\delta(z; x^k, \xi_{k+1})\| \leq (\kappa + \sqrt{3}\varepsilon)\|z^k - \bar{z}^k\| + 2\sqrt{3}\varepsilon\|z - z^k\|. \quad (20)$$

Step 2.2. We finally prove the inequality (5). Suppose that z^{k+1} is a KKT point of $P_{\text{cvx}}(x^k; \xi_{k+1})$, we have $0 \in \hat{\varphi}(z^{k+1}; x^k, \xi_{k+1}) + N_K(z^{k+1})$. This inclusion implies $\delta(z^{k+1}; x^k, \xi_{k+1}) \in \hat{\varphi}(z^{k+1}; \bar{x}^k, \xi_{k+1}) + N_K(z^{k+1}) \equiv L(z^{k+1}; \xi_{k+1})$ by the definition (16) of $\delta(z^{k+1}; x^k, \xi_{k+1})$. On the other hand, since $0 \in \hat{\varphi}(\bar{z}^k; \bar{x}^k, \xi_k) + N_K(\bar{z}^k)$, which is equivalent to $\delta_1 := M(\xi_{k+1} - \xi_k) \in L(\bar{z}^k; \xi_{k+1})$, applying (A3) we get

$$\begin{aligned}\|z^{k+1} - \bar{z}^k\| &\leq \gamma \|\delta(z^{k+1}; x^k, \xi_{k+1}) - \delta_1\| \\ &\leq \gamma \|\delta(z^{k+1}; x^k, \xi_{k+1})\| + \gamma \|M(\xi_{k+1} - \xi_k)\|.\end{aligned}$$

Combining this inequality and (20) with $z = z^{k+1}$ to obtain

$$\|z^{k+1} - \bar{z}^k\| \leq \gamma(\kappa + \sqrt{3}\varepsilon)\|z^k - \bar{z}^k\| + 2\sqrt{3}\gamma\varepsilon\|z^{k+1} - z^k\| + \gamma\|M(\xi_{k+1} - \xi_k)\|. \quad (21)$$

Using the triangular inequality, after a simple arrangement, (21) implies

$$\begin{aligned}\|z^{k+1} - \bar{z}^{k+1}\| &\leq \frac{\gamma(\kappa + 3\sqrt{3}\varepsilon)}{1 - 2\sqrt{3}\gamma\varepsilon}\|z^k - \bar{z}^k\| + \frac{1 + 2\sqrt{3}\gamma\varepsilon}{1 - 2\sqrt{3}\gamma\varepsilon}\|z^{k+1} - \bar{z}^k\| \\ &\quad + \frac{\gamma}{1 - 2\sqrt{3}\gamma\varepsilon}\|M(\xi_{k+1} - \xi_k)\|.\end{aligned} \quad (22)$$

Now, let us define $\omega_k := \frac{\gamma(\kappa + 3\sqrt{3}\varepsilon)}{1 - 2\sqrt{3}\gamma\varepsilon}$, $c_k := \frac{\gamma}{1 - 2\sqrt{3}\gamma\varepsilon} \left[\frac{2\sqrt{3}\gamma\varepsilon + 1}{1 - c_0\gamma} + 1 \right]$. By the choice of ε at *Step 1.1*, we can easily check that $\omega_k \in (0, 1)$ and $c_k > 0$. Substituting (15) into (22) and using the definitions of ω_k and c_k , we obtain

$$\|z^{k+1} - \bar{z}^{k+1}\| \leq \omega_k\|z^k - \bar{z}^k\| + c_k\|M(\xi_{k+1} - \xi_k)\|,$$

which proves (5). The theorem is proved. \square

If $\Gamma \equiv \{\xi\}$ then the RTSCP method collapses to the full-step SCP method described in Section 1. Without loss of generality, we can assume that $\xi_k = 0$ for all $k \geq 0$. The following corollary immediately follows from Theorem 1.

Corollary 1. *Suppose that $\{z^j\}_{j \geq 1}$ is the sequence of the KKT points of $P_{\text{cvx}}(x^{j-1}; 0)$ generated by the SCP method described in Section 1 and that the assumptions of Theorem 1 hold for $\xi_k = 0$. Then*

$$\|z^{j+1} - \bar{z}\| \leq \omega\|z^j - \bar{z}\|, \quad \forall j \geq 0, \quad (23)$$

where $\omega \in (0, 1)$ is the contraction factor. Consequently, this sequence converges linearly to a KKT point \bar{z} of $P(0)$.

4 Numerical example: control of an underactuated hovercraft

In this section we apply RTSCP to the control of an underactuated hovercraft. We use the same model as in [12], which is characterized by the following differential equations:

$$\begin{cases} m\ddot{y}_1(t) = (u_1(t) + u_2(t)) \cos(\theta), \\ m\ddot{y}_2(t) = (u_1(t) + u_2(t)) \sin(\theta), \\ I\ddot{\theta}(t) = r(u_1(t) - u_2(t)), \end{cases} \quad (24)$$

where $y(t) = (y_1(t), y_2(t))^T$ is the coordinate of the center of mass of the hovercraft (see Fig. 2); $\theta(t)$ represents the direction of the hovercraft; $u_1(t)$ and $u_2(t)$ are the fan

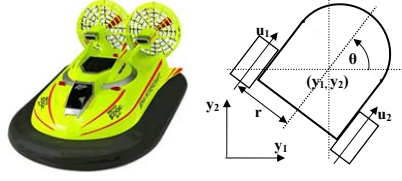


Fig. 2. RC hovercraft and its model [12].

thrusts; m and I are the mass and moment of inertia of the hovercraft, respectively; and r is the distance between the central axis of the hovercraft and the fans.

The problem considered is to drive the hovercraft from its initial position to the final parking position corresponding to the origin of the state space while respecting the constraints

$$\underline{u} \leq u_1(t) \leq \bar{u}, \quad \underline{u} \leq u_2(t) \leq \bar{u}, \quad \underline{y}_1 \leq y_1(t) \leq \bar{y}_1, \quad \underline{y}_2 \leq y_2(t) \leq \bar{y}_2. \quad (25)$$

To formulate this problem so that we can use the proposed method, we discretize the dynamics of the system using the Euler discretization scheme. After introducing a new state variable $\xi := (y_1, y_2, \theta, \dot{y}_1, \dot{y}_2, \dot{\theta})^T$ and a control variable $u := (u_1, u_2)^T$, we can formulate the following optimal control problem:

$$\begin{aligned} \min_{\substack{\xi_0, \dots, \xi_N \\ u_0, \dots, u_{N-1}}} & \sum_{n=0}^{N-1} [\|\xi_n\|_Q^2 + \|u_n\|_R^2] + \|\xi_N\|_S^2 \\ \text{s.t.} & \quad \xi_0 = \bar{\xi}, \\ & \quad \xi_{n+1} = \phi(\xi_n, u_n) \quad \forall n = 0, \dots, N-1, \\ & \quad (\xi_0, \dots, \xi_N, u_0, \dots, u_{N-1}) \in \tilde{\Omega}, \end{aligned} \quad (26)$$

where $\phi(\cdot, \cdot)$ represents the discretized dynamics and the constraint set $\tilde{\Omega}$ can be easily deduced from (25). By introducing a slack variable s and using the convex constraint:

$$s \geq \sum_{n=0}^{N-1} [\|\xi_n\|_Q^2 + \|u_n\|_R^2] + \|\xi_N\|_S^2, \quad (27)$$

we can transform (26) into $P(\bar{\xi})$ of a variable $x := (s, \xi_0^T, \dots, \xi_N^T, u_0^T, \dots, u_{N-1}^T)^T$ and the objective function $c^T x = s$. Note that $\bar{\xi}$ is an online parameter. It plays the role of ξ_k in the RTSCP algorithm along the moving horizon (see Section 2).

We implemented the RTSCP algorithm using a primal-dual interior point method for solving the convex subproblem $P_{\text{cvx}}(x^{k-1}; \bar{\xi}_k)$. We performed a simulation using the same data as in [12]: $m = 0.974\text{kg}$, $I = 0.0125\text{kg} \cdot \text{m}^2$, $r = 0.0485\text{m}$, $\underline{u} = -0.121\text{N}$, $\bar{u} = 0.342\text{N}$, $\underline{y}_1 = \underline{y}_2 = -2\text{m}$, $\bar{y}_1 = \bar{y}_2 = 2\text{m}$, $Q = \text{diag}(5, 10, 0.1, 1, 1, 0.01)$, $S = \text{diag}(5, 15, 0.05, 1, 1, 0.01)$, $R = \text{diag}(0.01, 0.01)$ and the initial condition $\xi_0 = \xi(0) = (-0.38, 0.30, 0.052, 0.0092, -0.0053, 0.002)^T$.

Figure 3 shows the results of the simulation where a sampling time of $\Delta t = 0.05\text{s}$ and $N = 15$ are used. The stopping condition used for the simulation is $\|y(t)\| \leq 0.01$.

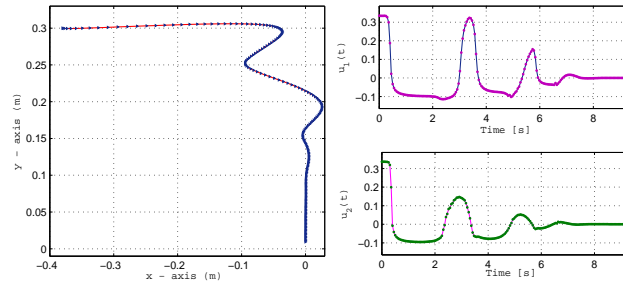


Fig. 3. Trajectory of the hovercraft after $t = 9.5$ s (left) and control input profile (right).

Acknowledgments. The authors would like to thank the anonymous referees for their comments and suggestions that helped to improve the paper.

This research was supported by Research Council KUL: CoE EF/05/006 Optimization in Engineering(OPTEC), GOA AMBioRICS, IOF-SCORES4CHEM, several PhD/postdoc & fellow grants; the Flemish Government via FWO: PhD/postdoc grants, projects G.0452.04, G.0499.04, G.0211.05, G.0226.06, G.0321.06, G.0302.07, G.0320.08 (convex MPC), G.0558.08 (Robust MHE), G.0557.08, G.0588.09, research communities (ICCoS, ANMMM, MLDM) and via IWT: PhD Grants, McKnow-E, Eureka-Flite+EU: ERNSI; FP7-HD-MPC (Collaborative Project STREP-grantnr. 223854), Contract Research: AMINAL, and Helmholtz Gemeinschaft: viCERP; Austria: ACCM, and the Belgian Federal Science Policy Office: IUAP P6/04 (DYSCO, Dynamical systems, control and optimization, 2007-2011).

References

1. L.T. Biegler: Efficient solution of dynamic optimization and NMPC problems. In: F. Allgöwer and A. Zheng (ed), *Nonlinear Predictive Control*, vol. 26 of *Progress in Systems Theory*, 219–244, Basel Boston Berlin, 2000.
2. L.T. Biegler and J.B. Rawlings: Optimization approaches to nonlinear model predictive control. In: W.H. Ray and Y. Arkun (ed), *Proc. 4th International Conference on Chemical Process Control - CPC IV*, 543–571. AICHE, CACHE, 1991.
3. H.G. Bock, M. Diehl, D.B. Leineweber, and J.P. Schlöder: A direct multiple shooting method for real-time optimization of nonlinear DAE processes. In: F. Allgöwer and A. Zheng (ed), *Nonlinear Predictive Control*, vol. 26 of *Progress in Systems Theory*, 246–267, Basel Boston Berlin, 2000.
4. M. Diehl: *Real-Time Optimization for Large Scale Nonlinear Processes*. vol. 920 of *Fortschr.-Ber. VDI Reihe 8, Meß-, Steuerungs- und Regelungstechnik*, VDI Verlag, Düsseldorf, 2002.
5. M. Diehl, H.G. Bock, and J.P. Schlöder: A real-time iteration scheme for nonlinear optimization in optimal feedback control. *SIAM J. on Control and Optimization*, 43(5):1714–1736, 2005.
6. M. Diehl, H.G. Bock, J.P. Schlöder, R. Findeisen, Z. Nagy, and F. Allgöwer: Real-time optimization and nonlinear model predictive control of processes governed by differential-algebraic equations. *J. Proc. Contr.*, 12(4):577–585, 2002.
7. M. Diehl, R. Findeisen, and F. Allgöwer: A stabilizing real-time implementation of nonlinear model predictive control. In: L. Biegler, O. Ghattas, M. Heinkenschloss, D. Keyes, and B. van Bloemen Waanders (ed), *Real-Time and Online PDE-Constrained Optimization*, 23–52. SIAM, 2007.
8. M. Diehl, R. Findeisen, F. Allgöwer, H.G. Bock, and J.P. Schlöder: Nominal Stability of the Real-Time Iteration Scheme for Nonlinear Model Predictive Control. *IEEE Proc.-Control Theory Appl.*, 152(3):296–308, 2005.
9. A. Helbig, O. Abel, and W. Marquardt: *Model Predictive Control for On-line Optimization of Semi-batch Reactors*. Pages 1695–1699, Philadelphia, 1998.
10. T. Ohtsuka: A continuation/GMRES method for fast computation of nonlinear receding horizon control. *Automatica*, 40(4):563–574, 2004.
11. S. M. Robinson: Strongly regular generalized equations. *Mathematics of Operations Research*, 5(1):43–62, 1980.
12. H. Seguchi and T. Ohtsuka: Nonlinear Receding Horizon Control of an Underactuated Hovercraft. *International Journal of Robust and Nonlinear Control*, 13(3–4):381–398, 2003.
13. V. M. Zavala and L.T. Biegler: The Advanced Step NMPC Controller: Optimality, Stability and Robustness. *Automatica*, 45:86–93, 2009.